Exponential ergodicity of branching processes with competition and immigration

李培森

Based on the joint work with Zenghu Li, Jian Wang, Xiaowen Zhou.

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Feller's branching diffusion

[Galton–Watson 1874] established branching process from i.i.d. $\mathbb{N}$-valued r.v. :

$$X(n) = \sum_{i=1}^{X(n-1)} \xi_{n,i}, \qquad n \geq 1.$$

Let $m = \mathbb{E}[\xi_{n,i}]$ and $2c = \mathrm{Var}[\xi_{n,i}]$. Then $(b := 1 - m)$

$$\begin{aligned} X(n) - X(n-1) &= \sum_{i=1}^{X(n-1)} (\xi_{n,i} - m) + (m-1)X(n-1) \\ &= \sqrt{2cX(n-1)} \sum_{i=1}^{X(n-1)} \frac{(\xi_{n,i} - m)}{\sqrt{2cX(n-1)}} - bX(n-1). \end{aligned}$$

A Feller's branching diffusion may arise as the scaling limit [Feller 1951]:

$$X_t = \lim_{k\to\infty} \frac{1}{k} X(\lfloor kt \rfloor), \qquad t \geq 0.$$

Feller's branching diffusion

The Feller's branching diffusion can be constructed as the solution to

$$x(t) = x + \int_0^t \sqrt{2cx(s)}\,\mathrm{d}B_s - \int_0^t bx(s)\,\mathrm{d}s,$$

where $b \in \mathbb{R}$ and $c, x \geq 0$. This equation is *weakly* equivalent to

$$x(t) = x + \int_0^t \int_0^{x(s)} W(\mathrm{d}s, \mathrm{d}u) - \int_0^t bx(s)\,\mathrm{d}s,$$

where $W(\mathrm{d}s, \mathrm{d}u)$ is a time-space Gaussian white noise based on $2c\mathrm{d}s\mathrm{d}u$.

The corresponding transition semigroup $(P_t)_{t\geq 0}$ satisfies the (branching property):

$$P_t(x,\cdot) * P_t(y,\cdot) = P_t(x+y,\cdot), \qquad x, y \in \mathbb{R}_+,$$

This means different individuals act independently.

CB-processes

In general, a continuous-state branching process (CB-process) is a càdlàg $[0,\infty]$-valued Markov process satisfying the branching property and it solves the stochastic equation [Bertoin–Le Gall 2006; Dawson–Li 2006/2012]:

$$X_t \;=\; X_0 + \int_0^t \int_0^{X_s} L(\mathrm{d}s, \mathrm{d}u).$$

Here $L(\mathrm{d}s, \mathrm{d}u)$ is a one-sided Lévy white noise given by

$$L(\mathrm{d}s, \mathrm{d}u) \;=\; W(\mathrm{d}s, \mathrm{d}u) - b\mathrm{d}s\mathrm{d}u + \int_0^1 z\tilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_1^\infty zM(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

where $M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is a Poisson random measure on $(0,\infty)^3$.

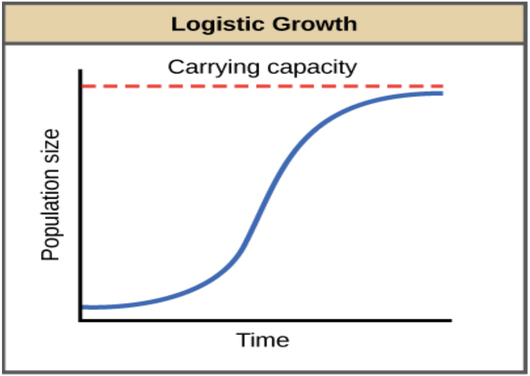
- $Z_t := Z_0 + \int_0^t \int_0^1 L(\mathrm{d}s, \mathrm{d}u)$ is a one-sided Lévy process.
- When $M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) \equiv 0$, the solution reduces to a Feller's branching diffusion.
- The process either goes to ∞ or goes to 0 as $t \to \infty$.

A deterministic model

A logistic growth model can be constructed as the solution to:

$$X_t = X_0 - \int_0^t (bX_s + dX_s^2)\mathrm{d}s,$$

where $-b > 0$ is the rate for each individual to reproduce offspring and $d > 0$ describes competition between each pair of individuals.



CBC-processes

[Berestycki et al. 2018] introduced the branching processes with competition

$$X_t = x + \int_0^t \int_0^{X_s} L(\mathrm{d}s, \mathrm{d}u) - \int_0^t g(X_{s-})\mathrm{d}s.$$

where g is a nondecreasing continuous function satisfying $g(0) = 0$.

- When $g(x) \equiv 0$, the solution reduces to a CB-process.
- When $g(x) = dx^2$, for some $d > 0$, the solution reduces to the logistic branching process introduced by [Lambert 2005].
- Since 0 is absorbing, it usually doesn't have a stationary distribution.

CBC-processes

The generator of a CBC-process is given by

$$Lf(x) = [-bx - g(x)]f'(x) + cxf''(x) + x\int_0^\infty [f(x+z) - f(x) - zf'(x)1_{\{z\le 1\}}]\mu(\mathrm{d}z).$$

Theorem [Lambert 2005]

Assume the following:

(1) $g(x) = dx^2$ for some $d > 0$;

(2)

$$\int_1^\infty \log z\mu(\mathrm{d}z) < \infty.$$

Then $\lim_{t\to\infty} X_t = 0$ a.s.

Question: How to construct a stationary model?

CBIC-processes

Let $\eta(t)$ be a subordinator given by

$$\eta(t) = \beta t + \int_0^t \int_0^\infty z N(\mathrm{d}s, \mathrm{d}z).$$

where $N(\mathrm{d}s, \mathrm{d}z)$ is a Poisson random measure on $(0, \infty)^2$ with intensity $\mathrm{d}s\nu(\mathrm{d}z)$. The continuous-state branching process with immigration and competition (CBIC-process) is defined as the solution to

$$X_t = \int_0^t \int_0^{X_s} L(\mathrm{d}s, \mathrm{d}u) - \int_0^t g(X_{s-})\mathrm{d}s + \eta_t, \tag{1}$$

The generator of a CBIC-process is given by

$$\begin{aligned} Lf(x) &= [-bx - g(x)]f'(x) + cxf''(x) \\ &\quad + x\int_0^\infty \big[f(x+z) - f(x) - zf'(x)1_{\{z\le 1\}}\big]\mu(\mathrm{d}z) \\ &\quad + \beta + \int_0^\infty \big[f(x+z) - f(x)\big]\nu(\mathrm{d}z). \end{aligned}$$

Branching processes with immigration and competition

The distribution property of a CBIC-process is completely determined by (Ψ, Φ, g).

- The branching mechanism Ψ is a function on $[0, \infty)$ with *Lévy–Khintchine type representation*:

$$\Psi(\lambda) = b\lambda + c\lambda^2 + \int_0^\infty \big(\mathrm{e}^{-\lambda z} - 1 + \lambda z 1_{\{z \le 1\}}\big)\mu(\mathrm{d}z),$$

  where $b \in \mathbb{R}$ and $c \ge 0$ are constants and $(1 \wedge z^2)\mu(\mathrm{d}z)$ is a finite measure on $(0, \infty)$.

- The immigration mechanism Φ is a function on $[0, \infty)$ with *Lévy–Khintchine type representation*:

$$\Phi(\lambda) = \beta\lambda + \int_0^\infty \big(1 - \mathrm{e}^{-z\lambda}\big)\nu(\mathrm{d}z),$$

  where $\beta \ge 0$ and $(1 \wedge z)\nu(\mathrm{d}z)$ is a finite measure on $(0, \infty)$.

- The competition mechanism g is a nondecreasing and continuous function on $[0, \infty)$ satisfying $g(0) = 0$.

Previous work

- Coupling method: Chen ('04, 05'), Chen and Wang ('93, 95')
- For Markov processes with additive Lévy noises: Luo–Wang ('16,'19), Schilling–Wang ('12), Majka ('17).
- For CBI-processes (without competition, mainly subcritical branching): Pinsky ('72), Li–Ma ('15), Friesen et al. ('20).
- For nonlinear CBI-processes: Li–Wang ('20).

Previous work

Consider the CBI-process ($g(x) \equiv 0$)

$$
\begin{aligned}
Lf(x) &= (\beta - bx)f'(x) + cxf''(x) + \int_0^\infty \big[f(x+z) - f(x)\big]\nu(\mathrm{d}z) \\
&\quad + x\int_0^\infty \big[f(x+z) - f(x) - zf'(x)1\!\!1_{\{z\leq 1\}}\big]\mu(\mathrm{d}z),
\end{aligned}
$$

for any $f \in C_b^2(\mathbb{R}_+)$.

Theorem [Pinsky 1972]

Assume that $-b + \int_1^\infty zm(\mathrm{d}z) < 0$. Then the CBI process is ergodic if and only if

$$\int_1^\infty \log z\nu(\mathrm{d}z) < \infty.$$

(If $-b + \int_1^\infty zm(\mathrm{d}z) > 0$, then the CBI-process is transient.)

Previous work

Theorem [Li–Ma 2015]

Suppose that $-b + \int_1^\infty z\mu(\mathrm{d}z) < 0$, $\nu(\mathrm{d}z) = 0, \beta > 0$ and

$$\int^{\infty-} \frac{\mathrm{d}\lambda}{\Psi(\lambda)} < \infty,$$

where

$$\Psi(\lambda) = b\lambda + c\lambda^2 + \int_0^\infty \big(\mathrm{e}^{-z\lambda} - 1 + z\lambda 1\!\!1_{\{z\le 1\}}\big)\mu(\mathrm{d}z), \quad \lambda \ge 0.$$

Then the CBI process is exponentially ergodic in total variation distance.

Remark:

- The integral condition doesn't hold when $b > 0$, $c = 0$ and $\mu(\mathrm{d}z) = 1\!\!1_{\{0<z\le 1\}} z^{-2}\mathrm{d}z$. In fact we have $\lim_{\lambda\to\infty} \frac{\lambda \log(1+\lambda)}{\Phi(\lambda)} = 1$.

Weighted total variation distance

We study the exponential ergodicity of the CBIC-process. For a suitable nonnegative function V, we define the matric on $\mathbb{R}_+$ by

$$d_V(x,y) = [2 + V(x) + V(y)]\mathbb{1}_{\{x \neq y\}}, \quad x, y \in \mathbb{R}_+. \tag{2}$$

The V-weighted total variation distance W_V is given by

$$\begin{aligned} W_V(\mu,\nu) &= \sup_{||f/(1+V)||\leq 1} \int_{\mathbb{R}_+} f(x)|\mu(\mathrm{d}x) - \nu(\mathrm{d}x)| \\ &= \inf_{\pi \in \mathscr{C}(\mu,\nu)} \int_{\mathbb{R}_+^2} d_V(x,y)\pi(\mathrm{d}x,\mathrm{d}y), \quad \mu,\nu \in \mathscr{P}(\mathbb{R}_+), \end{aligned}$$

where $\mathscr{C}(\mu,\nu)$ is the set of probabilities on $\mathbb{R}_+^2$ with marginals μ and ν.

The CBIC-process is said to be exponentially ergodic in W_V with rate $\lambda > 0$ if its transition semigroup $(P_t)_{t\geq 0}$ has a unique stationary distribution μ and

$$W_V(\mu, P_t(x,\cdot)) \leq C(x)\mathrm{e}^{-\lambda t}, \quad t \geq 0, x \geq 0.$$

Main results

Condition 1:

(i) The diffusion parameter $c > 0$ or there is a constant $\kappa > 0$ such that

$$\inf_{0 \leq x \leq \kappa} \mu \wedge (\delta_x * \mu)(\mathbb{R}_+) > 0.$$

(ii) There are a C^2-function $V : \mathbb{R}_+ \to \mathbb{R}_+$ and constants $\lambda, C > 0$ such that,

$$\lim_{x \to \infty} V(x) = \infty$$

and

$$LV(x) \leq -\lambda V(x) + C, \qquad x \in \mathbb{R}_+.$$

Theorem 1

Suppose that Condition 1 holds true. Then the CBIC-process is exponentially ergodic in W_V with rate λ'.

(The expression of λ' is very complex, so we omit it.)

Main results

The general result applies to the Lyapunov functions V_1 defined by

$$V_1(x) = x.$$

Theorem 2

Suppose that $\mu(\mathrm{d}z) = \sigma z^{-\alpha-1}\mathbb{1}_{\{z\geq 0\}}\mathrm{d}z$ for some $\sigma > 0$ and $\alpha \in (1,2)$. In addition, assume that

$$\int_1^\infty z\nu(\mathrm{d}z) < \infty$$

and

$$\liminf_{x\to\infty} \frac{g(x)}{x} > \frac{\sigma}{\alpha - 1} - b.$$

Then the CBIC-process is exponentially ergodic in the V_1-weighted total variation distance.

Main results

The general result applies to the Lyapunov functions $V_{\log}$ defined by

$$V_{\log}(x) = \log(1+x), \quad x \geq 0.$$

Theorem 3

Suppose that $\mu(\mathrm{d}z) = \sigma z^{-\alpha-1}\mathbb{1}_{\{z\geq 0\}}\mathrm{d}z$ for some $\sigma > 0$ and $\alpha \in (0,1]$. In addition, assume that

$$\int_1^\infty \log(1+z)\nu(\mathrm{d}z) < \infty$$

and

$$\begin{cases} \liminf\limits_{x\to\infty} \dfrac{g(x)}{x^{2-\alpha}} > \dfrac{\sigma\pi\sin(\alpha\pi)}{\alpha} & \text{for } 0 < \alpha < 1, \\ \liminf\limits_{x\to\infty} \dfrac{g(x)}{x\log x} > \sigma & \text{for } \alpha = 1. \end{cases}$$

Then the CBIC-process is exponentially ergodic in the $V_{\log}$-weighted total variation distance.

Remark: This condition is sharp in some sense.

Coupling method

An operator $\tilde{L}$ is called a *coupling operator* of L if for $h(x,y) = f(x) + g(y)$ for $f, g \in C_b^2(\mathbb{R}_+)$

$$\tilde{L}h(x,y) = Lf(x) + Lg(y) \qquad x, y \in \mathbb{R}_+.$$

The Markov process $(X_t, Y_t)_{t\geq 0}$ generated by $\tilde{L}$ is called coupling process.

The *coupling time* $T = \inf\{t \geq 0 : X_t = Y_t\}$ is the first time that the two marginal processes meet each other.

Notice that:

- $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ have the same transition semigroup $(P_t)_{t\geq 0}$.
- $\mathbb{P}(X_t = Y_t | t \geq T) = 1$.

Coupling method

- Setp1: Construct a Markov coupling $\{(X_t, Y_t) : t \geq 0\}$ of CBIC-processes;
- Step2: Find a nonnegative (nonsymmetric) control function $G_0(x, y) = \phi(x)f(x - y)$ such that

$$\tilde{L}G_0(x, y) \leq -\lambda G_0(x, y)$$

The function should control the distance d_V in the sense:

$$c_1 G_0(x, y) \leq d_V(x, y) \leq c_2 G_0(x, y), \quad (x, y) \in D.$$

- Step3: By Gronwall's Lemma, we have

$$W_V(P_t(x, \cdot), P_t(y, \cdot)) \leq K\mathrm{e}^{-\lambda t} d_V(x, y), \quad t \geq 0,$$

which immediately leads to the desired result.

Refined basic coupling

Refined basic coupling was established in [Luo and Wang 2019].

Similarly, in our situation, when $x > y \geq 0$, this coupling is described by

$$(x,y) \longrightarrow \begin{cases} (x+z, y+z+(x-y)), & \frac{1}{2} y \mu_{-(x-y)}(\mathrm{d}z), \\ (x+z, y+z-(x-y)), & \frac{1}{2} y \mu_{(x-y)}(\mathrm{d}z), \\ (x+z, y+z), & y\Big[\mu(\mathrm{d}z) - \frac{1}{2}\mu_{-(x-y)}(\mathrm{d}z) \\ & \qquad -\frac{1}{2}\mu_{(x-y)}(\mathrm{d}z)\Big] + \nu(\mathrm{d}z), \\ (x+z, y), & (x-y)\mu(\mathrm{d}z), \end{cases}$$

where $\mu_x = [\mu \wedge (\delta_x * \mu)](\mathrm{d}z)$.

Refined basic coupling

Let $W_0(\mathrm{d}s, \mathrm{d}u)$ be a Gaussian white noise based on $2c\mathrm{d}s\mathrm{d}u$. Let $M_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}v)$ be a Poisson random measure with intensity $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u\mathrm{d}v$ and compensated measure $\tilde{M}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}v)$. Let

$$\begin{aligned} L_0(\mathrm{d}s, \mathrm{d}u) &= W_0(\mathrm{d}s, \mathrm{d}u) - b\mathrm{d}s\mathrm{d}u + \int_{\{0<z\le 1\}}\int_{\{0<v\le 1\}} z\tilde{M}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}v) \\ &\quad + \int_{\{1<z<\infty\}}\int_{\{0<v\le 1\}} zM_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}v) \end{aligned}$$

and

$$\begin{aligned} L_0^*(\mathrm{d}s, \mathrm{d}u) &= -W_0(\mathrm{d}s, \mathrm{d}u) - b\mathrm{d}s\mathrm{d}u + \int_{\{0<z\le 1\}}\int_{\{0<v\le 1\}} z\tilde{M}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}v) \\ &\quad + \int_{\{1<z<\infty\}}\int_{\{0<v\le 1\}} zM_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}v). \end{aligned}$$

Refined basic coupling

For any $x \geq y \geq 0$ a Markov coupling $\{(X_t, Y_t) : t \geq 0\}$ of CBIC-processes is defined by the pathwise uniqueness solution to the system of stochastic equations:

$$X_t = x + \int_0^t \int_0^{X_{s-}} L_0(\mathrm{d}s, \mathrm{d}u) - \int_0^t g(X_s)\mathrm{d}s + \eta_t \tag{3}$$

and

$$\begin{aligned} Y_t &= y + \int_{t\wedge T}^t \int_0^{X_{s-}} L_0(\mathrm{d}s, \mathrm{d}u) - \int_0^t g(Y_s)\mathrm{d}s + \eta_t \\ &\quad + \int_0^{t\wedge T} \int_0^{X_{s-}} L_0^*(\mathrm{d}s, \mathrm{d}u) + \xi_t, \end{aligned}$$

where $T = \inf\{t \geq 0 : X_t \leq Y_t\} = \inf\{t \geq 0 : X_t = Y_t\}$ and $(0 \leq \rho_i, \gamma_i \leq 1)$

$$\begin{aligned} \xi_t &= \int_0^{t\wedge T} \int_0^\infty \int_0^{Y_{s-}} \int_0^{\rho_1(Y_{s-}-X_{s-},z)} (X_{s-} - Y_{s-}) M_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}v) \\ &\quad + \int_0^{t\wedge T} \int_0^\infty \int_0^{Y_{s-}} \int_{\rho_1(Y_{s-}-X_{s-},z)}^{\rho_2(X_{s-},Y_{s-},z)} (Y_{s-} - X_{s-}) M_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u, \mathrm{d}v). \end{aligned}$$

Thank you!